Research Guru
Online Journal of Multidisciplinary Subjects (ISSN : 2349-266X)
www.researchguru.net
Volume-15, Issue-3, December-2021

Kantowski-Sachs Cosmological Model with a Minimally Coupled Scalar Field

DINKAR SINGH CHAUHAN
Department of Mathematics, Mahadev Post Graduate College, Bariyasanpur, Varanasi-221112, India
E-mail : dschauhan9616@gmail.com

R. S. SINGH
Department of Mathematics, Post Graduate College, Ghazipur-233001, India
E-mail : rss1968@rediffmail.com

Abstract: *The present study deals with spatially homogeneous and anisotropic Kantowski-Sachs Cosmological model with a minimally coupled scalar field in Einstein's theory of gravitation. To get a determinate model of the universe, we assume that the scalar expansion (θ) of the model is proportional to the shear (σ). This condition leads to $A = \alpha B^m$, where α and m are constants. Various physical and geometical properties of the model have also been discussed.*

Keywords: *Kantowski-Sachs space-time, scalar field, anisotropic models.*

1. Introduction

Usually, it is assumed that the universe is filled with a perfect fluid. But observations suggest that cosmological dynamics cannot be fully explained by this standard matter. The observational results lead to the search of some kinds of exotic matter which would generate sufficient negative pressure to drive the late-time cosmic acceleration. One such exotic matter is the scalar field which provides the necessary negative pressure causing acceleration [6,21]. Thus, the scalar field cosmological models are of great importance in the study of the early universe, particularly in the investigation of inflation. The recent discovery of cosmic acceleration [1,13,17,19,22] has stimulated the interest to study cosmological models based on scalar fields. The cosmological models based on scalar fields have been investigated by Guth [12] and Frieman et. al. [11] to explain the possible early inflationary scenarios as well as the dark matter problem.

The dynamics of the evolution of the universe is often realized by scalar field with a proper scalar potential. The self-interacting potential can act as an effective cosmological constant which derives a period of inflation. It depends on the specific form of the potential as a function of scalar field. The scalar field with an exponential potential is a strong candidate for dark matter in spiral galaxies and is consistent with observations of current accelerated expansion of the universe [15,16]. Many authors [5,7,20] have studied scalar field cosmological models with an exponential scalar potential within general relativity. Several researchers [2-4,8-10,14] have studied the scalar field cosmology in Friedmann-Roberston-Walker (FRW) model with different forms of scalar potentials like flat, constant and exponential potentials. Recently, Singh et al. [23] have studied minimally coupled scalar field cosmology in anisotropic cosmological model.

Motivated by the above discussions, in this paper, we have studied Kantowski-Sachs cosmological model with a minimally coupled scalar field in general theory of relativity. The outline of the paper is as follows. The metric and the field equations are presented in Section 2. Section 3 deals with the solutions of the field equations. In Section 4, we describe some physical and geometric properties of the model. Finally, conclusions are given in section 5.

2. The Metric and Field Equations

we consider the homogeneous and anisotropic space-time described by Kantowski-Sachs metric in the form

$$ds^2 = dt^2 - A^2 dr^2 - B^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{1}$$

where the metric potentials A and B are functions of cosmic time t alone.

Kantowski-Sachs class of metric represents homogeneous but anisotropically expanding (contracting) cosmologies and provides models where the effect of anisotropy can be estimated and compared with FRW class of cosmologies [24].

In the case of gravity minimally coupled to a scalar field potential $V(\phi)$, the Lagrangian L is

$$L = \int \left[R - \frac{1}{2} g^{ij} \phi,_i \phi,_j - V(\phi) \right] \sqrt{(-g)}\, d^4x, \tag{2}$$

which on variation of L with respect to dynamical fields lead to Einstein's field equations (in gravitational units $8\pi G = c = 1$)

$$R_{ij} - \frac{1}{2} R g_{ij} = -T_{ij} \tag{3}$$

with energy-momentum tensor

$$T_{ij} = \phi,_i \phi,_j - \left[\frac{1}{2} \phi,_k \phi^{,k} + V(\phi) \right] g_{ij} \tag{4}$$

and $\phi^i_{;i} = -\dfrac{dV(\phi)}{d\phi}$, (5)

where comma and semicolon denotes ordinary and covariant differentiation respectively. The function ϕ depends on cosmic time *t* only due to homogeneity.

In co-moving coordinate system, we have from (4)

$$T_1^1 = T_2^2 = T_3^3 = -\left[\frac{\dot{\phi}^2}{2} + V(\phi)\right]$$

and $$T_4^4 = \frac{\dot{\phi}^2}{2} - V(\phi). \tag{6}$$

The Einstein's field equations (3) with he help of (6) for the metric (1) are given by

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{1}{B^2} = -\left[\frac{\dot{\phi}^2}{2} + V(\phi)\right], \tag{7}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -\left[\frac{\dot{\phi}^2}{2} + V(\phi)\right], \tag{8}$$

$$2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{1}{B^2} = \frac{\dot{\phi}^2}{2} - V(\phi), \tag{9}$$

and the equation (5) for the scalar field takes the form

$$\ddot{\phi} + \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\dot{\phi} + \frac{d}{d\phi}V(\phi) = 0. \tag{10}$$

Here the overhead dot denotes differentiation with respect to cosmic time *t*.

We define the following parameters to be used in solving Einstein's field equations for the metric (1).

The average scale factor S, the volume scale factor V and the generalized mean Hubble's parameter H are defined as

$$S = \left(AB^2\right)^{\frac{1}{3}}, \tag{11}$$

$$V = S^3 = AB^2, \tag{12}$$

$$H = \frac{\dot{S}}{S} = \frac{1}{3}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right). \tag{13}$$

An important observational quantity is the deceleration parameter q, which is defined as

$$q = -\frac{S\ddot{S}}{\dot{S}^2}. \tag{14}$$

The expansion scalar θ, the shear scalar σ and the average anisotrophy parameter A_m are defined as

$$\theta = u^i_{;i} = \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}, \tag{15}$$

$$\sigma = \sqrt{\frac{1}{2}\sigma_{ij}\sigma^{ij}} = \frac{1}{\sqrt{3}}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right), \tag{16}$$

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2. \tag{17}$$

3. Solution of the Field Equations

We try to solve the field equations by choosing an additional relation in the form of some physical condition signifying some particular scenario. For spatially homogeneous metric, the normal congruence to the homogeneous hyper surface satisfies the condition $\frac{\sigma}{\theta} =$ constant. This condition leads to

$$A = \alpha B^m, \tag{18}$$

where α is an integrating constant and m > 1.

Subtracting (8) from (7) and using (18) in the resulting equation, we obtain

$$2\ddot{B} + 2(m+1)\frac{\dot{B}^2}{B} = \frac{2}{(m-1)B}. \tag{19}$$

Let $\dot{B} = f(B)$ which implies that $\ddot{B} = ff'$, where $f' = \frac{df}{dB}$.

Hence, equation (19) leads to

$$\frac{d\left(f^2\right)}{dB} + 2(m+1)\frac{f^2}{B} = \frac{2}{(m-1)B}. \tag{20}$$

Integrating, equation (20), we obtain

$$f^2 = \left(\frac{dB}{dt}\right)^2 = \left[\frac{1}{(m^2-1)} + \frac{l}{B^{2(m+1)}}\right], m > 1, \tag{21}$$

where l is a constant of integration.

Using (18) and (21), the line element (1) reduces to

$$ds^2 = \left[\frac{1}{(m^2-1)} + \frac{l}{B^{2(m+1)}}\right]^{-1} dB^2 - \alpha^2 B^{2m} dr^2 - B^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{22}$$

The metric (22) can be transformed through a proper choice of coordinates to the form

$$ds^2 = \left[\frac{1}{(m^2-1)} + \frac{l}{T^{2(m+1)}}\right]^{-1} dT^2 - \alpha^2 T^{2m} dr^2 - T^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{23}$$

4. Some Physical and Geometrical Properties

The expressions for the scalar of expansion (θ), shear scalar (σ), the average Hubble's parameter (H), deceleration parameter (q), proper volume (V) and the average anisotropy parameter A_m for the model (23) are respectively given by

$$\theta = (m+2)\left[\frac{1}{(m^2-1)T^2}+\frac{l}{T^{2(m+2)}}\right]^{\frac{1}{2}}, \tag{24}$$

$$\sigma = \frac{(m-1)}{\sqrt{3}}\left[\frac{1}{(m^2-1)T^2}+\frac{l}{T^{2(m+2)}}\right]^{\frac{1}{2}}, \tag{25}$$

$$H = \frac{(m+2)}{3}\left[\frac{1}{(m^2-1)T^2}+\frac{l}{T^{2(m+2)}}\right]^{\frac{1}{2}}, \tag{26}$$

$$q = \frac{3}{(m+2)}\left[\frac{1}{(m^2-1)T^2}+\frac{l}{T^{2(m+2)}}\right]^{-1}\left[\frac{1}{(m^2-1)T^2}+\frac{(m+2)l}{T^{2(m+2)}}\right]-1, \tag{27}$$

$$V = \alpha T^{m+2} \tag{28}$$

$$A_m = 2\left(\frac{m-1}{m+2}\right)^2. \tag{29}$$

The scalar field (ϕ) and the scalar field potential V(ϕ) are, respectively given by

$$\phi = \int\left[\frac{2(m^2-1)(2m+1)l+2mT^{2(m+1)}}{(m^2-1)lT^2+T^{2(m+2)}}\right]^{\frac{1}{2}} dT+\phi_o, \tag{30}$$

where ϕ_o is an integrating constant.

$$V(\phi) = -\left(\frac{m}{m-1}\right)\frac{1}{T^2}. \tag{31}$$

To find determinate solution of equation (30), we assume $l = 0$, which yields

$$\phi = \sqrt{2m}\log T+\phi_o. \tag{32}$$

In this case, the scalar field potential V(ϕ) takes the form

$$V(\phi) = -\left(\frac{m}{m-1}\right)e^{-\sqrt{\frac{2}{m}}(\phi-\phi_o)}. \tag{33}$$

The mathematical analysis of derived Kantowski-Sachs model are as follows

- It is observed that the spatial volume is zero and the expansion scalar is infinite at T = 0, which shows that the universe starts evolving with zero volume with infinite rate of expansion. The universe exhibits an initial singularity of POINT type at T = 0.
- The physical quantities θ, σ, and H start off with extremely large values and continuously decrease with increase of time and tend to zero as $T \to \infty$.
- From equation (27), we observe that as $T \to 0$, $q \to 2$ (> 0), therefore the model decelerates in the standard way. It deserves mention that the decelerating models are also consistent with the recent CMB observations made by WMAP as well as the high redshift supernovae Ia data including SN199ff at Z = 1.7559 [25]. Also, $T\to\infty$ implies that $q\to -1$ (<0) which shows that the present universe is accelerating. This facts is supported by the recent observations of SNe Ia [17-19].
- We observe that $\lim_{T\to\infty} \frac{\sigma}{\theta} = \frac{(m-1)}{\sqrt{3}(m+2)} \neq 0(as\ m>1)$, the model never approach to isotropy at any time.
- The mean anisotropy parameter is uniform throughout the whole expansion of the universe.
- We observe that at the beginning i.e. at T = 0, $\phi \to -\infty$. It increases with time and tends to $+\infty$ at $T \to \infty$. Thus, the Kinetic energy vanishes at the end of the evolution (an infinite expansion). For negative ϕ, the potential $V(\phi)$ is unbounded from below, one might expect that one can always construct negative energy solutions.

5. Conclusions

We have presented Kantowski-Sachs cosmological model with a minimally coupled scalar field in Einstein's theory of gravitation. The study showed that the universe was decelerating in the past and accelerating at present time. At the beginning of the evolution i.e. at T = 0, the scalar field $\phi \to -\infty$. During the evolution, ϕ increases and at the end of the evolution (an infinite expansion) when $T \to \infty$, it tends to $+\infty$. The Kinetic energy vanishes at $T\to \infty$. The potential $V(\phi)$ start off with extremely large negative value and tends to zero at late-time. For large values of T, the scalar of expansion and shear scalar tend to zero but initially both are infinite. The spatial volume becomes infinite and the physical as well as dynamical quantities tend to zero.

The model has a point type singularity at T = 0 and does not approach the isotropy for large value of T.

References

[1] Ade, P.A.R. et. al. (2014). Planck 2013 results. XVI. cosmological parameters, Astron. Astrophys. 571, **A16** (66pp).

[2] Barrow, J.D. (1987). Cosmic no hair theorems and inflation, Phys. Lett. B **187**, 12-16.

[3] Barrow, J.D. Saich, P. (1993). Scalar-field cosmologies, Class. Quant. Grav. **10**, 279-283.

[4] Barrow, J.D. Parsons, P. (1995). Inflationary models with logarithmic potentials, Phys. Rev. D **52**, 5576-5587.

[5] Burd, A.B. and Barrow, J.D. (1998). Inflationary models with exponential potential, Nucl. Phys. B **308**, 929-945.

[6] Coldwell, R.R. Dave, R. Steinhardt, P.J. (1998). Cosmological imprint of an energy component with general equation of state, Phys. Rev. Lett. **80**, 1582-1585.

[7] Chimento, L.P. (1998). General solution to two-scalar field cosmologies with exponential potentials, Class. Quantum Grav. **15**, 965-974.

[8]Coley, A.A. Ibanez, J. and Van den Hoogen, (1997). Homogeneous scalar field cosmologies with an exponential potential, J. Math. Phys. **38**, 5256-5271.

[9] Copeland, E.J. Mizuno, S. and Shaeri, M. (2009). Dynamics of a scalar field in Roberston-Walker spacetimes, Phys. Rev. D **79**, 103515 (10pp).

[10] Ellis, G.F.R. and Madsen, M.S. (1991). Exact scalar field cosmologies, Class. Quant. Grav. **8**, 667-676.

[11] Fireman, J.A. and Waga, I. (1998). Constraints from high redshift supernovae upon scalar field cosmologies, Phys. Rev. D **57**, 4642-4650.

[12] Guth, A.H. (1981). Inflationary universe: A possible solution to the horizon and flatness problems, Phys. Rev. D **23**, 2(10pp)

[13] Hinshaw, G. et. al. (2013). Nine year Wilkinson Microwave Anisotropy Probe (WMAP) observations: cosmological parameters results, Astrophys. J. Suppl. **208**, 19(25pp).

[14] Heard, I.P.C. and Wands, D. (2002). Cosmology with positive and negative exponential potentials, Class. Quant. Grav. **19**, 5435-5448.

[15] Liddle, A.R. and Lyth, D. (2000). Cosmological Inflation and Large Scale Structure (Cambridge University Press).

[16] Mukhanov, V. (2005). Physical Foundation of Cosmology (Cambridge University Press).

[17] Perlmutter, S. et. al. (1999). Measurement of Omega and Lambda from 42 High-Redshift Supernovae, Astrophys. J. **517**, 565 - 586.

[18] Perlmutter, S. et al. (1998). Discovery of a supernova explosion at half the age of the universe, Nature **391**, 51-54.

[19] Riess, A.G. et.al. (1998). Observational evidence from supernovae for an accelerating universe and a cosmological constant, Astron. J. **116**, 1009 1038.

[20] Russo, J.G. (2004). Exact solution of scalar field cosmology with exponential potential and transient acceleration, Phys. Lett. B **600**, 185-190.

[21] Sahni, V. (2004). 5 dark matter and dark energy, Lect. Notes Phys. **653**, 141-179.

[22] Suzuki, N. et. al. (2012). The space Hubble telescope cluster supernovae survey. V. improving the dark energy constraints above z>1 and building an early-type-Hosted supernova sample, Astrophys. J. **746**, 85(24pp).

[23] Singh, C.P. and Srivastava, M. (2017). Minimally coupled scalar field cosmology in anisotropic cosmological model, Pramana J. Phys. **88**, 22(10pp).

[24] Thorne, K.S. (1967). Primordial element formation, primordial magnetic fields, and the isotropy of the universe, Astrophys. J. **149**, 51-68.

[25] . Vishwakarma, R.G. (2003). Is the present expansion of the universe realy accelerating, Mon. Not. R. Astron. Soc. **345**, 545-551.